

Tensors, Quantum Entanglement & Optimization

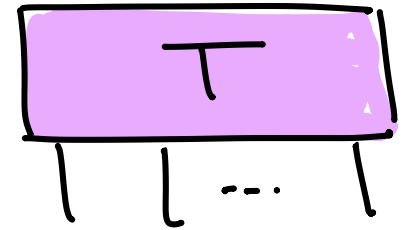
Michael Walter (LMU)

TENORS @Trento, July 2026



Tensors as Quantum States

$$|T\rangle \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_d}$$

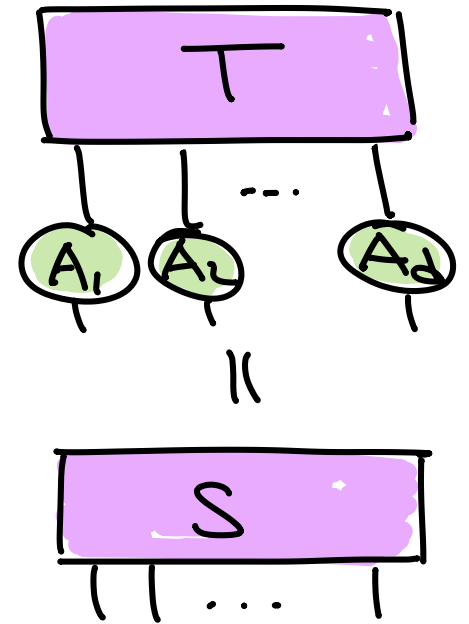


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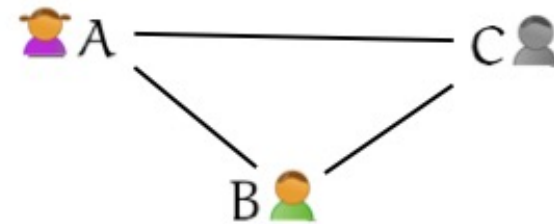
"SLOCC", "filtering",
"restriction"

$$S = (A_1 \otimes \dots \otimes A_d) T$$



Natural operations that "do not create entanglement"

tensor factors
 $\cong$
 q. subsystems/
 particles/parties/...

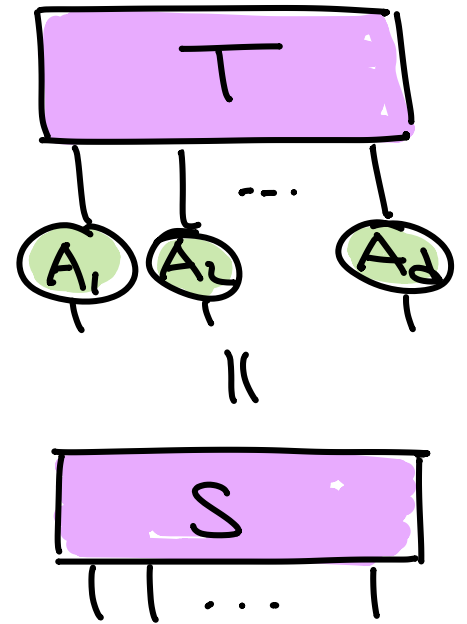


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Natural operations that "do not create entanglement"

Can also allow limits: $S \in \overline{G \cdot T}$ "degeneration"

Where $G = GL(n_1) \times \dots \times GL(n_d)$

Q: Normal forms? Conversion rules? Algos?

Normal Forms of Quantum States

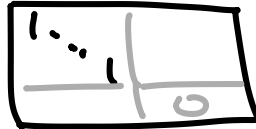
$$G = GL(n_1) \times \dots \times GL(n_d)$$

$d=2$: Gaussian elim. $\rightarrow$  $\hat{=}$ Singular values $(\underbrace{1, \dots, 1}_{\text{rank}})$

$d > 2$?

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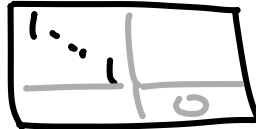
$d > 2$? Consider quantum marginals

$$G_j = \begin{array}{c} \boxed{S_j} \\ \boxed{\dots \mid \dots} \\ \boxed{S_j^*} \end{array} = S_j S_j^* \text{ where } S_j : \mathbb{C}^{n_j} \times \dots$$

j^{th} principal flattening

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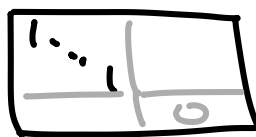
j^{th} principal flattening

Proposal (Verstraëte et al, 2001):

- $S \in \overline{G \cdot T}$ is in normal form if $\sigma_j = I \ (\forall j)$
unique up to $K := U(n_1) \times \dots \times U(n_d)$

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- Also:

$$S \leftarrow (I \otimes \dots \otimes I \otimes \sigma_j^{-1/2} \otimes I \otimes \dots \otimes I) S \quad (\forall j)$$

"Flip Fbp", "Tensor Scaling"

Normal Forms of Quantum States (cont.) $G = G_L(n_1) \times \dots \times G_L(n_d)$

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Analysis (Burgisser - ... - W-Widgerson, 2016): To find tensor s.t. all $\sigma_j \approx_\epsilon I$, $\text{poly}(\text{input size}, \frac{1}{\epsilon})$ iterations suffice.

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key observation: iterations decrease

$$F(g_1, \dots, g_d) = \left\| \left(\bigotimes_{\substack{\det(g_j) = 1}} \det(g_j)^{-1/n_j} g_j \right) \cdot T \right\|^2$$

... and $\inf F > 0$ if T semistable

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AM/GM inequality!

... and $\inf F > 0$ if T semistable

make quantitative using invariant theory!

$$\begin{aligned} F_{\text{new}} &= \det(\sigma_j)^{1/n_j} \cdot n_j \\ &\leq \text{tr}(\sigma_j) = F_{\text{old}} \end{aligned}$$

Normal Forms of Quantum States (cont.) $G = G_L(n_1) \times \dots \times G_L(n_d)$

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det = 1

AM/GM inequality!

... and $\inf F > 0$ if T semistable

generic tensors

have normal form ∞

Same algo tests

semistability

Tensors & Norm Minimization

Q: Can we similarly optimize norms for other actions?
(And is it good for anything?) **A:** YES oo

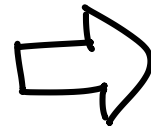
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① Tensors as Data in Statistics:

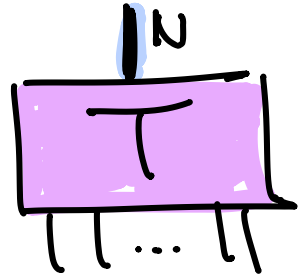
$$T_1, T_2, \dots \in \mathbb{R}^{n_1 \times \dots \times n_d}$$

Samples



$$T = (T_j)$$

$$\mathbb{R}^{n_1} \otimes \dots \otimes \mathbb{R}^{n_d} \otimes \mathbb{R}^N$$



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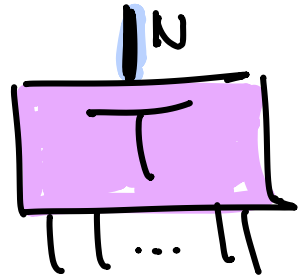
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Easy to see:

max likelihood in
"tensor normal model"
 $\mathcal{N}(0, \Sigma_1 \otimes \dots \otimes \Sigma_d)$

$$\equiv \text{min. norm} \quad \|(G_1 \otimes \dots \otimes S_d \otimes I) T\|^2$$

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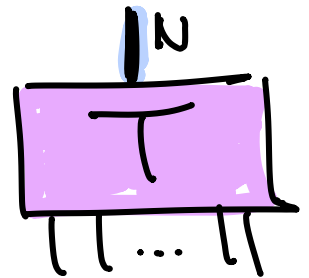
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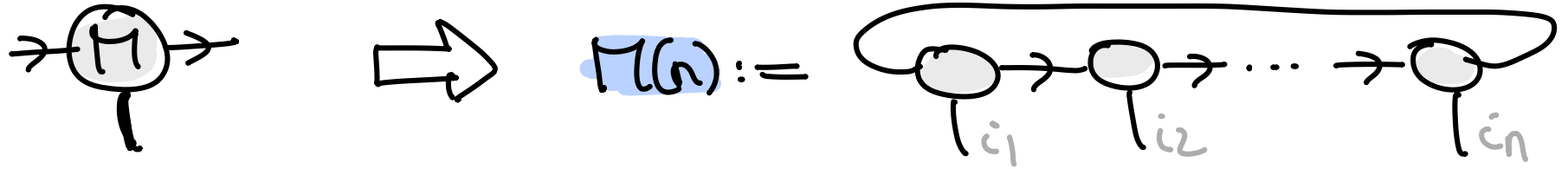
$$\stackrel{\text{min. norm}}{=} \|(G_1 \otimes \dots \otimes G_d \otimes I) T\|^2$$

↳ "Same" flip-flop algo
Computes MLE ∞

see Améndola et al,
Franks - ... W

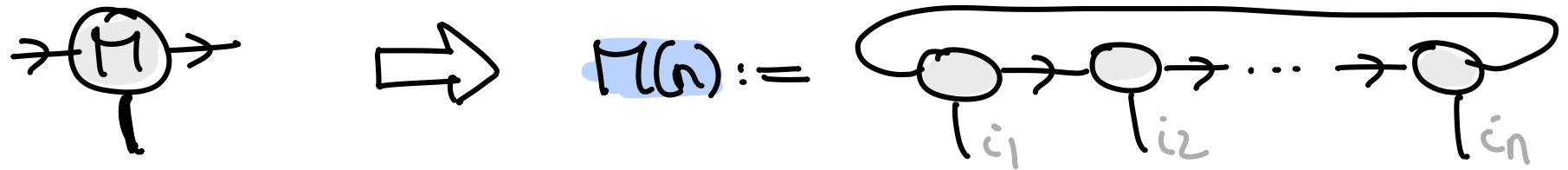
② Tensors Define Tensor Networks:

(Uniform) Matrix Product States:



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Invariant polys for "Simultaneous Conjugation":

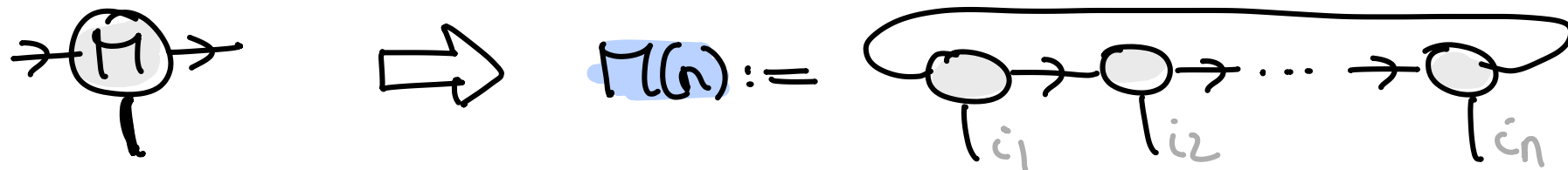
$$GL(n) \hookrightarrow (\mathbb{C}^{n \times n})^{\oplus N}, \quad g \bullet (M_j) = (g M_j g^{-1})$$



(new) Canonical form by
minimizing $\|g \bullet M\|^2$
+ fundamental theorem

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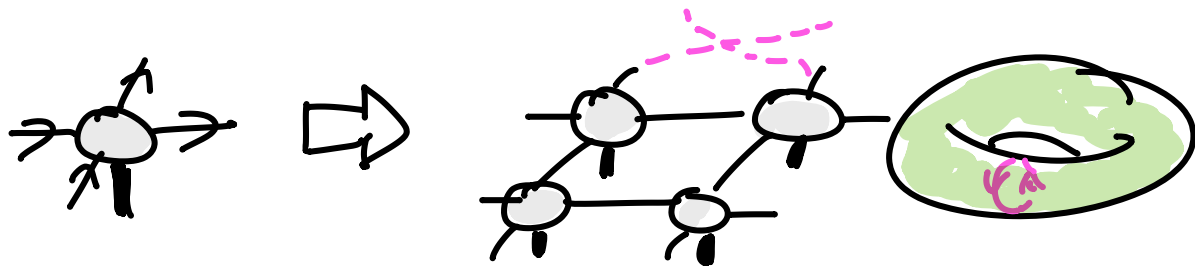
Invariant polys for "Simultaneous conjugation":

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→ generalizes to high dimensions
(but with a twist) PETS

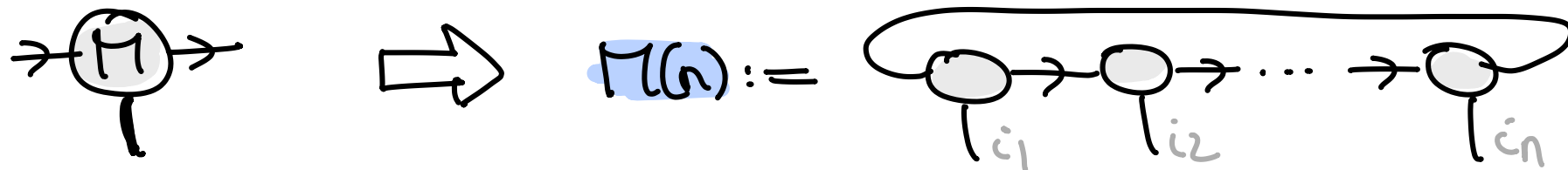


see Acuniva...-W

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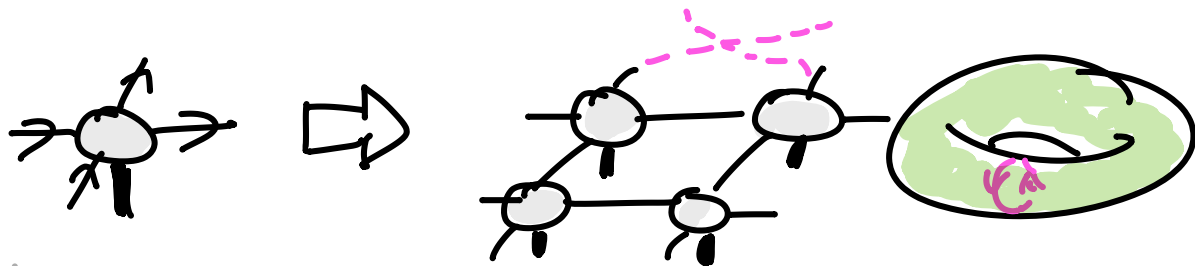


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② Semistability testing
& GIT $\cong$ symplectic quotient...

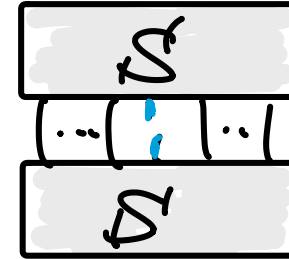
see Aravira-...-W

Back to Quantum States & Marginals

Recall: • Tensor action: $S = (g_1 \otimes \dots \otimes g_d)^T$

• Quantum marginals

$$Q_j =$$



Q: Which marginals
can be achieved?

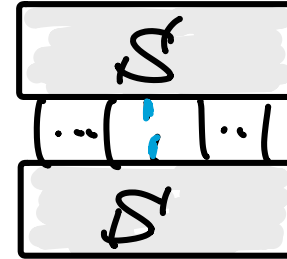
$\longleftrightarrow$ Normal forms for
unstable T ?

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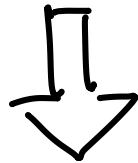
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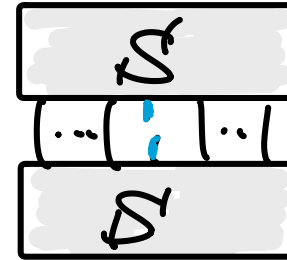
$$\{(\sigma_1, \dots, \sigma_d) : S \in \overline{G \cdot T}, \|S\| = 1\} = ?$$

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$$\sigma_j =$$



Q: Which marginals can be achieved?

Normal forms for unstable T ?



Entanglement Polytope:

(W...-Christandl, 2002)

$$\Delta(T) = \{ (\text{spec } \sigma_1, \dots, \text{spec } \sigma_d) : S \in \overline{G \cdot T}, \|S\| = 1 \}$$

Wild but true: It's a polytope!

← "hard" proof
(Mumford, Kirwan)

Entanglement Polytopes

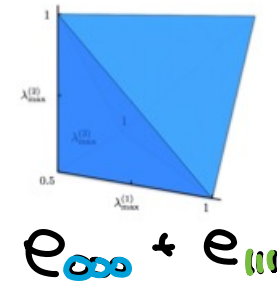
$$G = GL(n_1) \times \dots \times GL(n_d)$$

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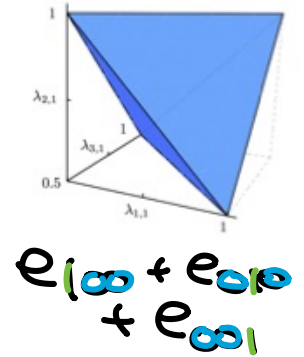
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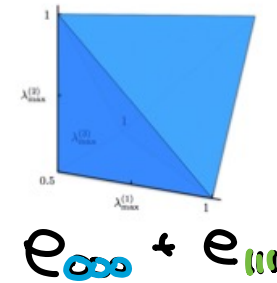
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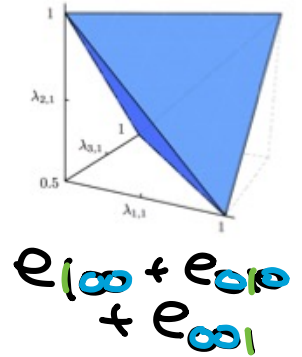
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(a "quantum marginal problem")

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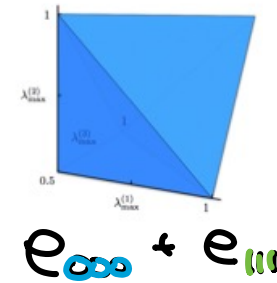
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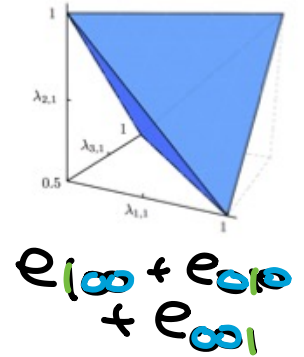
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Properties: • $(\frac{1}{n_1}, \dots, \frac{1}{n_d}) \in \Delta(T) \iff$ semistable $\searrow$

has Vestrak et al
normal form

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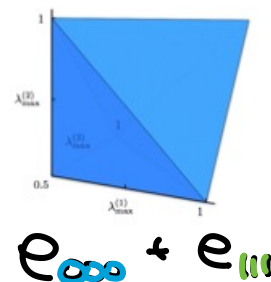
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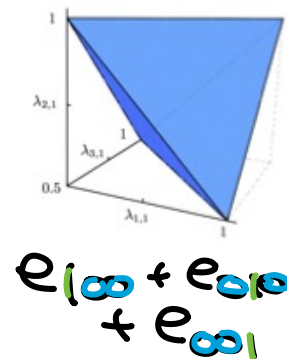
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Properties: • $(\frac{1}{n_1}, \dots, \frac{1}{n_d}) \in \Delta(T) \iff$ semistable $\searrow$

• Invariant: $\Delta(T) = \Delta(S \cdot T)$

has Verstraëke et al
normal form

• Monotonic: $S \in \overline{G \cdot T} \implies \Delta(S) \subseteq \Delta(T)$

} and generalise
normal form

↳ get "entanglement measures" by max. over $\Delta(T)$! 7/17

Tensors as Computational Problems

multiplication of
r numbers

$$\triangle_r = \sum_{j=1}^r e_j \otimes e_j \otimes e_j$$

vs

$n \times n$ matrix multiplication

$$\begin{array}{c} n \quad n \\ \diagdown \quad \diagup \\ n \end{array} = \sum_{i,j,k=1}^n e_{ij} \otimes e_{jk} \otimes e_{ik}$$

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reduction between problems $\hat{=}$ restriction/degeneration of tensors

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$$\frac{\begin{array}{c} n \text{ (green)} \quad n \text{ (orange)} \\ \text{---} \\ n \text{ (blue)} \end{array}}{n} = \sum_{i,j,k=1}^n e_{ij} \otimes e_{jk} \otimes e_{ik}$$

reduction between problems $\hat{=}$ restriction/degeneration of tensors

$\hookrightarrow$ algebraic complexity of matrix multiplication captured by:

$$\omega := \lim_{n \rightarrow \infty} \min \left\{ \frac{\log r}{\log n} : \triangle_r \longrightarrow \frac{\begin{array}{c} n \text{ (green)} \quad n \text{ (orange)} \\ \text{---} \\ n \text{ (blue)} \end{array} \right\}$$

Interpretation:

$O(n^\omega)$ operations suffice to
multiply two $n \times n$ matrices

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$$= \lim_{n \rightarrow \infty} \min \left\{ \frac{k}{\ell} : \triangle_2^{\otimes k} \longrightarrow \left(\frac{\triangle_2}{2} \right)^{\otimes \ell} \right\}$$

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Tensor Parameters & Entanglement Polytope

Quantum Functionals (Christandl et al, 2017): For $\theta \in \mathbb{R}_{\geq 0}^d, \sum \theta_j = 1$

$$F_{\theta}(T) := 2^{E_{\theta}(T)} \quad \text{where} \quad E_{\theta}(T) := \max_{x \in \Delta(T)} \sum_{j=1}^d \theta_j H(x_j)$$

Entanglement
polytope

Shannon entropy:
 $H(p) = \sum_i p_i \cdot \log \frac{1}{p_i}$

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Wild, but true: Multiplicative w.r.t. $\otimes$, additive w.r.t. $\oplus$ ← "hard" proof
 (+ normalized + monotone)

↳ bounds asymptotic conversion rates:

$$T^{\otimes k} \rightarrow S^{\otimes \ell} \quad \Rightarrow \quad \frac{k}{\ell} \geq \frac{E_{\Theta}(S)}{E_{\Theta}(T)}$$

cf. talk
by Angelo

$$\hookrightarrow \text{rank}(T) \geq F_{\Theta}(T) \geq \text{subrank}(T)$$

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Several recent **tensor parameters** (= entanglement measures) are defined by **convex/concave optimization** over **entanglement polytope** !

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↓
Q. Functionals
entropy

↓
G-stable rank (DedSai)
related to $\|\cdot\|_\infty$

↓
NC rank
 $\|\cdot\|_1$

↓
 ℓ^2 -norm
also for $\mathfrak{g}$ on action

Tensor Parameters & Entanglement Polytope

Several recent tensor parameters (= entanglement measures) are defined by convex/concave optimization over entanglement polytope !

↓
Q. Functionals
entropy

↓
G-stable rank (Dedekind)
related to $\|\cdot\|_\infty$

↓
NC rank
 $\|\cdot\|_1$

↓
 ℓ^2 -norm
also for $\mathfrak{g}$ on action

Results (Doğan-Sakabe-W, 2026):

- Efficient algorithms for all these: $\text{poly}(\frac{1}{\varepsilon}, \text{input size})$
- For q. functionals, "entropic tensor scaling"

$$\mathcal{S} \leftarrow (\sigma_1^{-\Theta_1/2} \otimes \dots \otimes \sigma_d^{-\Theta_d/2}) \mathcal{S}$$

Most interesting if T NOT generic (otherwise computes WF). 10/17

Tensor Parameters & Entanglement Polytope

Several recent tensor parameters (= entanglement measures) are defined by convex/concave optimization over entanglement polytope !

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- Structural insights: simple proofs that T_Θ is $\otimes$ & $\oplus$

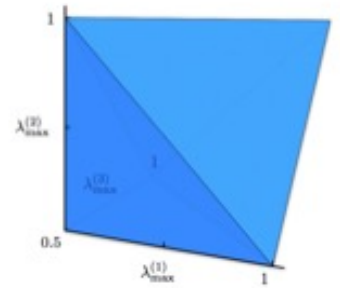
$S = S' \otimes S''$ is preserved by algo $\Rightarrow T_\Theta(S) \leq T_\Theta(S') T_\Theta(S'')$
 $\geq$ easy

Most interesting if T NOT generic (otherwise computes WF). 10/17

Optimization on Polytopes

Classical problem:

$$\left[\begin{array}{l} \min \Phi(x) \\ x \in \triangle \\ \text{polytope} \end{array} \right] \quad \text{Convex}$$

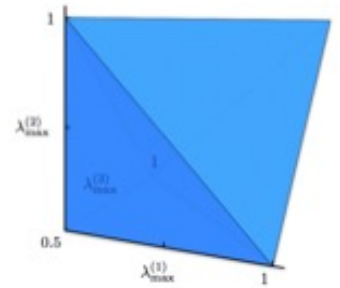


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"explicitly"
if Δ given as vertices/facets:
Standard methods such as ellipsoid



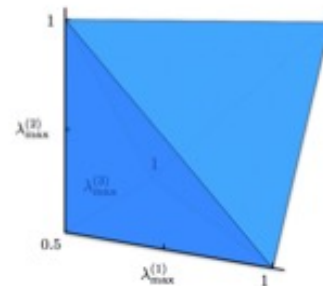
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if $\triangle$ implicitly defined,
Can sometimes still solve,
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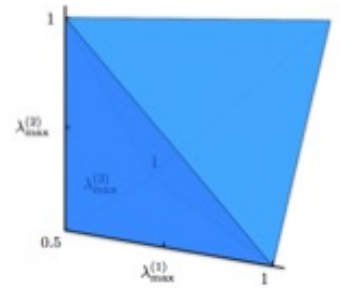
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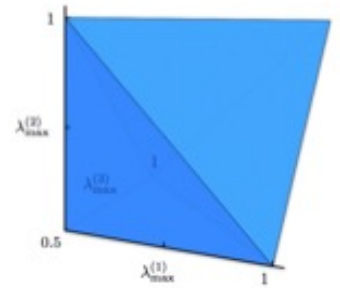
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Example:

$$\triangle = \frac{\text{conv} \{ \omega_j \}}{= \{ \nabla f(x) \}}$$

for $f: \mathbb{R}^n \rightarrow \mathbb{R}$
 $f(x) = \log \left[\sum_j e^{x \cdot \omega_j} \right]$

↳ polytope succinctly encoded as "gradient set" of convex f ! 11/17

Polytopes from Group Actions

Kirwan, Ness,
... Brieskorn...-W

Can similarly associate polytopes to any "nice" group action?

E.g. entanglement polytopes:

$$G = GL(n_1) \times \dots \times GL(n_d) \hookrightarrow V = \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_d}$$

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To any $T \in V$, associate fn. on $P = PD(n_1) \times \dots \times PD(n_d)$:

$$f: P \rightarrow \mathbb{R}, \quad f(p) = \log \underbrace{\langle T, (p_1 \otimes \dots \otimes p_d) T \rangle}_{= \|\sqrt{p} \cdot T\|^2}$$

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$$= \|\sqrt{p} \cdot T\|^2$$

Easy to verify:

$$\nabla f(p) = \bar{p}^{1/2} (\sigma_1, \dots, \sigma_d) p^{-1/2}$$

where σ_j q. marginals of $S = \sqrt{p} \cdot T_{/1\dots 1}$ ☺

$$\hookrightarrow \Delta(T) = \overline{\{ \text{spec } \sqrt{p} \nabla f(p) \sqrt{p} : p \in P \}}$$

12/17

Entanglement Polytope from Group Action

Almost a "gradient set":

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☹️ f is NOT convex
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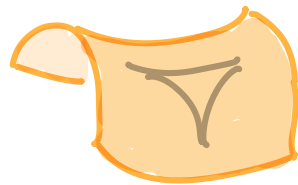
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😊 f is convex along geodesics of P as Riemannian manifold:

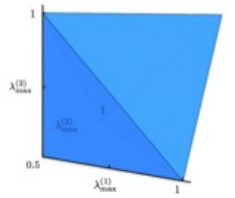
$$\gamma(t) = \sqrt{p} e^{Xt} \sqrt{p}$$



😊 $\sqrt{p} \cdot \sqrt{p}$ is parallel transport to I

Likewise for any G . If commutative: $P \cong \mathbb{R}^n$, $f \cong \log\text{-sum-exp}$. 13/17

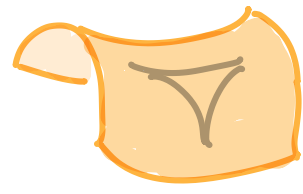
Optimization on Entanglement Polytope



$$\min_{x \in \Delta(\Gamma)} \Phi(x)$$

e.g. $\Phi = \sum$ Shannon entropy

Optimization on Entanglement Polytope



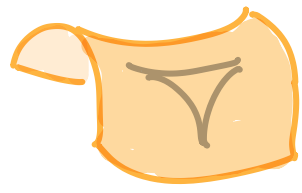
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$$= \inf_{p \in \mathcal{P}} \Phi(\text{spec } \sqrt{p} \nabla f(p) \sqrt{p})$$

just saw this

Optimization on Entanglement Polytope



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unitarily-invariant lift
(convex if Φ symmetric)

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What steps to take to minimize? WLOG $p = I$. Then:

$$\nabla(\dots \text{objective} \dots) = \nabla^2 f(I) [\nabla Q(\nabla f(I)), \bullet]$$

≥ 0 because f convex $\Rightarrow$ good descent direction!

Cf. Hirai's flow

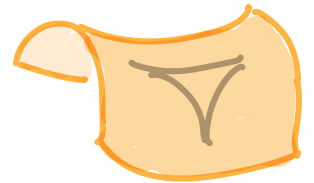
Mirror Descent on Entanglement Polytopes

(and other moment polytopes)

Summary: To minimize convex Φ over entanglement polytope, lift to unitarily-invariant convex fn. Q & apply iteration:

$$p \leftarrow \sqrt{p} e^{-\eta \nabla Q(\sqrt{p} \nabla f(p) \sqrt{p})} \sqrt{p}$$

suitable step size



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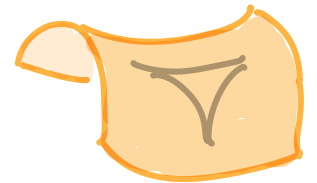
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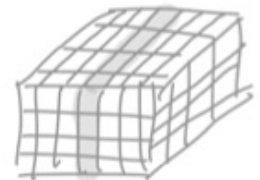
$$\Sigma \leftarrow e^{-\frac{\eta}{2} \nabla Q(\sigma_1, \dots, \sigma_d)} \cdot \Sigma$$

substep size

q. marginals



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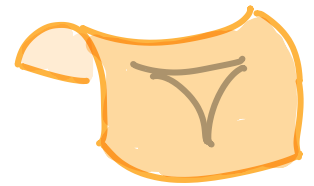
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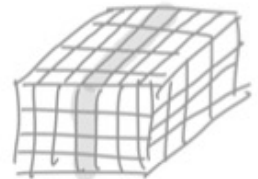
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easy-to-check criterion:
quantitative convergence
for explicit step size

Can be understood as version of
"mirror descent" to Hadamard setting

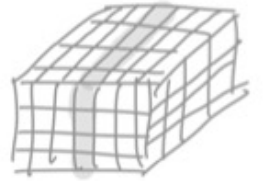
perspective also helped resolve
question by Stollen

15/17

Mirror Descent on Entanglement Polytopes

Concretely: To minimize Φ , run

$$S \leftarrow e^{-\frac{\eta}{2} \nabla Q(\sigma_1, \dots, \sigma_d)} \cdot S$$



Example: $\Phi = \frac{1}{2} \|\cdot\|_2^2$

Q. Functionals:

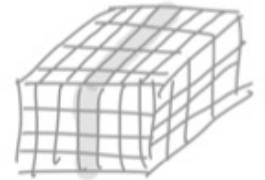
$$\Phi = \sum_j \theta_j \cdot H(x_j)$$

Shannon
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Q. Functions:

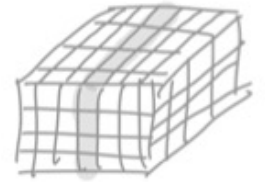
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- ordinary gradient descent
- unlike flip-flop, also converges for unstable T

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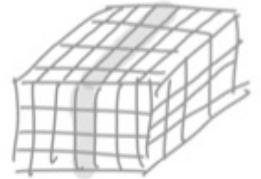
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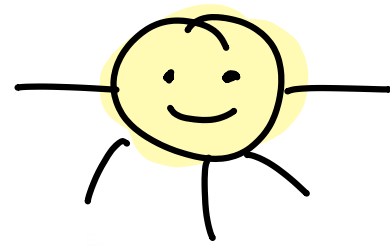
quantum entropy

$\hookrightarrow \nabla Q(\sigma_1, \dots, \sigma_d) \approx$
 $(\theta_1 \cdot \log \sigma_1, \dots, \theta_d \cdot \log \sigma_d)$

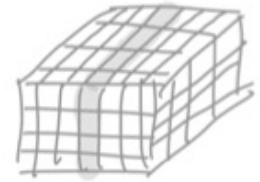
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Phew ☺

Summary & Outlook

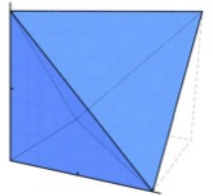
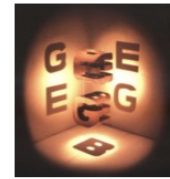


Tensors appear in many different roles

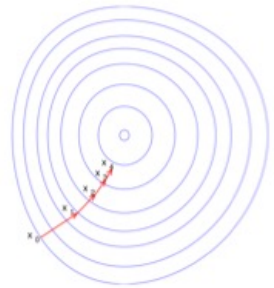


q. information, algebraic problems, statistical samples...

Symmetries / group actions create links
and give interesting problems & quantities..



...that can be tackled effectively using **convex optimization**
on curved spaces



Thanks for Your Attention !

...along with some
algebraic geometry
invariant theory
∞