Approximate QCAs and a converse to the Lieb-Robinson bounds

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joint work with Daniel Ranard (MIT) and Freek Witteveen (Amsterdam)

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Motivation

Quantum cellular automata model strictly local dynamics. However:

**Lieb-Robinson**: Local Hamiltonian evolution obeys *approximate* light cone.

For short-range interactions, there is Lieb-Robinson velocity $v$ such that support of local operators grows as $vt$, up to exponential tails.

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Physics question: Are local dynamics generated by local Hamiltonians?
  ▶ That is, can we find converse to Lieb-Robinson bounds?
  ▶ How about lattice translations?
  ▶ Boundary dynamics generated by bulk local Hamiltonian?

Mathematics question: Classify approximately local dynamics.

Our results: Approximately local dynamics in 1D have structure & index theory similar to QCAs. In particular, obtain a converse to LR bounds.
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Quantum Cellular Automata
Setup: Infinite spin chains

\[ \mathbb{C}^d \]

\[ n \]

\[ X \]

It is convenient to work in the Heisenberg picture:

\[ A_n = \text{Mat}(d) \quad \sim \quad A_X = \bigotimes_{n \in X} A_n \quad \sim \quad A_{\text{loc}} = \bigcup_{X \in \Lambda} A_X \]

Quasi-local \( C^* \)-algebra:

\[ A = \overline{A_{\text{loc}}} \| \cdot \| = \bigotimes_{n \in \mathbb{Z}} A_n \]

We can also define \( A_{\geq n} = A_{\{n, n+1, \ldots\}} \subseteq A \), etc.

Local dynamics are naturally modeled by automorphisms \( \alpha: A \to A \).
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Quantum cellular automata (QCAs)

Quasi-local algebra on infinite 1D lattice:

\[ \mathcal{A} = \bigotimes_{n \in \mathbb{Z}} \mathcal{A}_n, \quad \mathcal{A}_n = \text{Mat}(d) \]

An automorphism \( \alpha: \mathcal{A} \to \mathcal{A} \) is a quantum cellular automaton (QCA) or locality preserving unitary (LPU) with radius \( R > 0 \) if:

\[ \alpha(\mathcal{A}_n) \subseteq \mathcal{A}_{\{n-R,...,n+R\}} \]

That is, the support of any local operator grows by at most \( R \):

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Classification of QCAs in 1D

Examples:

Theorem (Gross-Nesme-Vogts-Werner, GNVW):
- Any QCA is a composition of circuit and shift.
- Shift cannot be implemented by circuit.
- QCAs modulo circuits are classified by quantized index.
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Examples:

Quantum circuit

Shift

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\text{index} = \text{amount of quantum information flowing right} - \text{amount of quantum information flowing left}
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This intuition can be made precise...

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How to define the index for shift QCAs:

\[ \text{index} = \log d_1 \]

\[ \text{index} = -\log d_2 \]
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(Re)defining the index

Cut chain in halves and consider corresponding Choi state $\rho_{LRL'R'}$. Then:

$$\text{index } \alpha = \frac{1}{2} \left( I(L : R') - I(L' : R) \right)$$

where $I(A : B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$ is the quantum mutual information.

Properties:

- quantized: index $\alpha \in \mathbb{Z}[\log p_i]$, $p_i =$ prime factors of local dimension
- additive: $\text{index } \alpha \otimes \beta = \text{index } \alpha + \text{index } \beta$
- robust: if $\alpha \approx \beta$ then $\text{index } \alpha = \text{index } \beta$
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Approximately Locality-Preserving Unitaries
Approximately locality-preserving unitaries (ALPUs)

Idea: Replace strict locality $\rightarrow$ Lieb-Robinson type bounds.

An automorphism $\alpha: \mathcal{A} \rightarrow \mathcal{A}$ is an approximately locality preserving unitary (ALPU) with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\forall b \in \mathcal{A}_X: \exists c \in \mathcal{A}_{r\text{-Neighborhood}(X)}: \|\alpha(b) - c\| \leq f(r)\|b\|$$
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**Useful notation:** $\mathcal{B} \subseteq_{\varepsilon} \mathcal{C}$ means

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**Examples:** QCAs, local Hamiltonian dynamics (Lieb-Robinson!), ...?
Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics...
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]
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  But: Mutual information defn. applies! Does index remain quantized?
Our results: Classification of ALPUs

ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
≡ QCAs *modulo* circuits

Theorem:

▶ ALPUs are classified by index that is quantized, additive, robust:

▶ Any ALPU is composition of quasi-local Hamiltonian dynamics and shift.
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Proof Ideas
A first attempt

Suppose we have an ALPU:

\[ \alpha \]

For any fixed \( n \), can truncate tails to obtain approximate morphism

\[ A_n \to A_{\{n-r,...,n+r\}}. \]

By a version of Ulam stability, can even find exact such morphism nearby.

However, for different sites \( n \), the images of these morphisms need *not* commute \( \to \) unclear how to patch together!

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Main tool: Stability of inclusion

**Theorem (Christensen, 80s):** If \( \mathcal{B} \subseteq \varepsilon \mathcal{C} \) for hyperfinite von Neumann algebras and \( \varepsilon < \frac{1}{8} \), then there is a unitary \( u \in (\mathcal{B} \cup \mathcal{C})'' \) such that

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u \mathcal{B} u^* \subseteq \mathcal{C} \quad \text{and} \quad \| u - I \| \leq 12 \varepsilon.\]

We extend this to show that, moreover:

- If \( x \in _\delta \mathcal{B} \) and \( x \in _\delta \mathcal{C} \), then \( \| x - uxu^* \| = O(\delta \| x \|) \).
- If \( x \in _\delta \mathcal{B}' \) and \( x \in _\delta \mathcal{C}' \), then \( \| x - uxu^* \| = O(\delta \| x \|) \).

Applied to ALPU, can localize image of any region, while preserving tails.
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How to use this?

**Key idea:** For any fixed cut, can apply unitaries near identity to construct automorphism that looks like QCA near this cut:

Left and right are decoupled – stronger than what we had before!
This allows us to glue different $\alpha_n, \alpha_{n+2}, \ldots$ together.

**Approximation Theorem:** For any 1D ALPU $\alpha$, there are QCAs $\beta_r$ of radius $2r$ such that $\beta_r \rightarrow \alpha$ strongly. In fact, if $f(r)$ are the tails of $\alpha$,

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\| (\alpha - \beta_r)_X \| \leq C_f f(r) \frac{\text{diam}(X)}{r}.
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![Diagram showing automorphism near a cut](image)

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![Diagram](image)

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By Christensen’s theorem, we can find unitary $u \approx I$ s.th.

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We can visualize this as above...
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and both algebras are in $\mathcal{A}_{\geq n-1}$. Thus, the latter contains unitary $\nu$ s.th.

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Key fact: Second unitary does not destroy locality achieved in first step!
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2. Why can we glue?

Compare two such local QCAs:

We can glue the red and the blue morphism by applying a unitary

\[ u \in \mathcal{A}_{n+1, n+2}. \]

Inductively we obtain a QCA.
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Index of an ALPU

Thus we proved that any ALPU $\alpha$ in 1D can be approximated by sequence of QCAs $\beta_r$ (sufficiently fast). This allows us to define the index:

$$\text{index } \alpha := \lim_{r \to \infty} \text{index } \beta_r$$

- well-defined, independent of choice of $\{\beta_r\}$
- inherits properties of GNVW index: quantized, additive, continuous, ...

If $O(\frac{1}{r^{1+\delta}})$-tails, can also compute as mutual information difference:

$$\text{index } \alpha = \frac{1}{2} \left( I(L : R') - I(L' : R) \right)$$
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How to obtain time-dependent quasi-local Hamiltonians?

- Start with ALPU of index $\alpha = 0$.
- Approximate $\alpha$ by QCA $\beta_1$ of same index. Thus $\beta_1$ is circuit and can be implemented by time-dependent local Hamiltonian evolution.
- Repeat with $\beta_1^{-1} \alpha$.

For an appropriate “schedule”, obtain time-dependent Hamiltonian

$$H(t) = \sum_{X} H_X(t)$$

that is piecewise constant and has geometrically local interactions

$$\|H_X(t)\| = O(f(k) \log k) \quad \text{with} \quad |X| = k \leq k(t).$$
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Summary and outlook

Approximately locality preserving unitaries (ALPUs) in 1D have structure & index theory generalizing the one of QCAs. In particular, implies a converse to Lieb-Robinson bounds. Main techniques are stability results for near inclusions of algebras. Many open problems:

▶ Periodic chain in 1D?
▶ Extension to high dimensions? 2D within reach...
▶ Beyond automorphisms: Is there a QCA near any “noisy” QCA?
▶ Other applications of stability results in QI?

Thank you for your attention!
Discussion slides