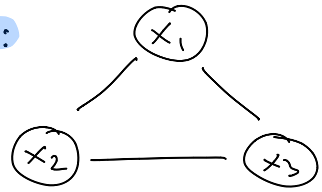
# Representation Theory Primer

§1 SAGAN

"solve" symmetries in math. problems ("add LA ☺")

Kirillov: 


$x' = \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{T} x$

Many steps? Easy, but how about $3 \to n$?

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$G$ finite group

**G-module / representation:** vector space (fin. dim. / $\mathbb{C}$) $V$ + group hom $\rho : G \to GL(V)$

* notation: $g \cdot v := \rho(g) v$
* if choose basis: "matrix representation": group hom $X : G \to GL_d$

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If $G$ acts on ($G \to S_M$) set $M$ ("$G \circlearrowright M$"): permutation rep $\mathbb{C}M$ (name clear!):

basis $\{e_x\}_{x \in M}$ & $\boxed{g \cdot e_x = e_{gx}}$

* $G \circlearrowright G$ by left multiplication $\leadsto$ left regular rep $\mathbb{C}[G]$
* $S_n \circlearrowright \{1, \ldots, n\}$: defining rep $\mathbb{C}\{1, \ldots, n\}$

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**Submodule** (inv. subspace): $W \subseteq V$ s.th. $gW \subseteq W$ $(\forall g)$

$V$ **reducible**: $\exists$ nontrivial submodule $W$ (i.e., $W \neq \{0\}, V$) $\qquad X = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$

Ex: $\mathbb{C}\{1,2,3\} \supseteq \mathbb{C}(e_1 \oplus e_2 \oplus e_3)$ Id submodule, irreducible

$V$ **decomposable**: $\exists$ nontrivial submodules $W, W'$: $V = W \oplus W'$ $\qquad X = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$

**Thm:** $G$ finite, $V$ rep / $\mathbb{C}$: ① $V$ reducible $\Rightarrow$ $V$ decomposable

② Maschke: $\exists k \geq 0$, $W_1, \ldots, W_k$ irreducible: $V = W_1 \oplus \cdots \oplus W_k$ "completely reducible"

Pf: ① Need to show ($\Rightarrow$). CRUCIAL FACT:

$\left.\begin{array}{l} W \subseteq V \text{ submodule} \\ \langle \cdot, \cdot \rangle \text{ inv. inner product} \end{array}\right\} \Rightarrow W^\perp$ submodule

And inv. inner products always exist: $\langle v, w \rangle := \sum_{g \in G} (gv, gw)$ (any inner prod.)

② Decompose until done. $\square$

**ex:** $\mathbb{C}\Pi$ carries inv. inner product s.th. $\langle e_x, e_y \rangle = \delta_{xy}$

↳ $\mathbb{C}\{1,2,3\} = \mathbb{C}(e_1+e_2+e_3) \oplus \text{span}\{e_1-e_2,\ e_2-e_3\}$ irreducible

<u>Aside</u> on complete reducibility:

* Still true if $\mathbb{C} \rightsquigarrow \mathbb{F}$ with char $\nmid \#G$. **BUT**: $G = \mathbb{Z}/2\mathbb{Z}$: $\mathbb{F}_2[G]$ ↯
* Still true if $G$ cpct. **BUT**: $\mathbb{Z} \to GL_2$, $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ ↯

How to compare reps?

**Homomorphism** ("intertwiner") between reps $V, W$:

$\Phi: V \to W$ linear s.th. $\Phi(g\cdot v) = g\cdot\Phi(w)$ "$\Phi g_V = g_W \Phi$"

* $\Phi$ bijective $\Rightarrow \Phi^{-1}$ hom. ("isomorphism", "equivalence", "$V \cong W$")
* $\ker(\Phi)$, $\text{im}(\Phi)$ submodules!

> <u>Lem</u> (Schur): $V, W$ irreducible, $\Phi: V \to W$ hom
> ① $\Phi \neq 0 \Rightarrow \Phi$ iso
> ② If $V = W / \mathbb{C}$: $\Phi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$

<u>Pf</u>: ① $\ker \Phi = \{0\}$ & $\text{im}(\Phi) = W$.

② NB: $\Phi - \lambda\cdot I$ is hom $\forall \lambda$.

Choose $\lambda$ = an eigenvalue of $\Phi$. Then: $\Phi - \lambda\cdot I$ NOT iso $\overset{①}{\Rightarrow}$ $\Phi - \lambda\cdot I = 0$. □

**ex:** $T = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} : \mathbb{C}\{1,2,3\} \to \mathbb{C}\{1,2,3\}$ is hom.

$\mathbb{C}\{1,2,3\} = \mathbb{C}(e_1+e_2+e_3) \oplus \text{span}\{e_1-e_2,\ e_2-e_3\}$

irred. ⟷ NOT iso ⟷ irred.

Schur $\overset{②}{\Rightarrow}$ $T \cong \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1/2 & \\ & & -1/2 \end{pmatrix}$, $T^n x \to \frac{x_1+x_2+x_3}{3}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$.

$T\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\0\end{pmatrix}$