1. **Joint typicality (1 point):** Let $X$ be random bit with $\Pr(X = 0) = \frac{1}{4}$, and let $Y$ be the output obtained by sending $X$ through a binary symmetric channel with bit flip probability $f = \frac{1}{4}$.

(a) Write down the joint distribution $P(x, y)$ and the marginal distributions $P(x)$ and $P(y)$.

(b) Consider the following two options:

- $x^N = 111010110101111$, $y^N = 110101111010100$
- $x^N = 111101011010111$, $y^N = 101101110110011$

One of the two pairs $(x^N, y^N)$ is in the jointly typical set $J_{16, \varepsilon}$ for $\varepsilon = 0.1$. Which one?

2. **Entropy inequalities and chain rule (1 point):** In this problem you can practice using entropy inequalities and the chain rule for the conditional entropy. Let $X^N$ be a random string of length $N$ with joint distribution $P(x_1, \ldots, x_N)$.

(a) Show that $H(X^N) \leq H(X_1) + \sum_{i=2}^{N} H(X_i|X_{i-1}) \leq \sum_{i=1}^{N} H(X_i)$.

Now let $Y^N$ denote the output of a memoryless channel $Q(y|x)$ when we input the string $X^N$. Thus, the joint distribution of $(X^N, Y^N)$ is given by $P(x^N, y^N) = P(x^N)Q(y_1|x_1) \cdots Q(y_N|x_N)$.

(b) Show that $H(Y_i|X^N Y_{i-1}) = H(Y_i|X_i)$ for $i = 1, \ldots, N$.

(c) Deduce that $I(X^N : Y^N) \leq \sum_{i=1}^{N} I(X_i : Y_i) \leq NC(Q)$.

*Hints: In the exercise class you proved that $H(Z|XY) \leq H(Z|Y)$ for any three random variables. Moreover, equality holds if and only if $X \rightarrow Y \rightarrow Z$ is a Markov chain. Use this in parts (a) and (b). In part (c), start by rewriting the mutual information so that you can apply the chain rule.*

3. **Random codes and typical set decoding (1 point):**

In this problem, you will generate a random code, implement the typical set decoder discussed in class, and study its performance for the binary symmetric channel. To get started, open the notebook at https://colab.research.google.com/github/amsqi/iit21-homework/blob/master/05-homework.ipynb and follow the instructions.

Please submit both the notebook and a PDF printout, or provide a link to your solution on Colab. You can achieve the maximum score if your solution produces the correct output. We will only have a closer look at your code in case of problems.