Algebraic and geometric complexity theory

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Agenda

1. Algebraic Complexity Theory
2. Geometry
3. Metapolynomials and Representation Theory

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2. Geometry
3. Metapolynomials and Representation Theory

Disclaimer

Common approaches to algebraic complexity theory:

- From a computer algebra perspective.
- From the perspective of emulating Boolean concepts/results in the algebraic world.

I take Valiant's approach, adjusted to geometric complexity theory.

Waring rank (also called symmetric rank)

Notation: $\mathbb{C}[x_1, \dots, x_k]_d = \mathrm{Sym}^d \mathbb{C}^k$.

Let $f \in \mathrm{Sym}^d \mathbb{C}^k$.

The Waring rank (also called symmetric rank) $\mathrm{WR}(f)$ of f is the smallest n such that

$$f = \sum_{i=1}^{n} (a_{i,1} x_1 + \cdots + a_{i,k} x_k)^d$$

for scalars $a_{i,j} \in \mathbb{C}$.

$$M_m \; = \; \sum_{i=1}^{m} x_{i,j}\, x_{j,k}\, x_{k,i} \qquad \in \mathrm{Sym}^3 \mathbb{C}^{m^2}$$

The matrix multiplication exponent:

$$\omega = \lim_{m \to \infty} \log_m(\mathrm{WR}(M_m))$$

[Chiantini-Hauenstein-I-Landsberg-Ottaviani 2017]

Iterated matrix multiplication

Let $f \in \mathrm{Sym}^d \mathbb{C}^k$.

The **complexity (or width)** $\mathrm{w}(f)$ is defined as the smallest n such that

$$f = (A_{1,1}x_1 + \ldots + A_{1,k}x_k) \cdots (A_{d,1}x_1 + \ldots + A_{d,k}x_k)$$

with $\forall 1 \le j \le k$:
- $A_{1,j} \in \mathbb{C}^{1\times n}$,
- $A_{i,j} \in \mathbb{C}^{n\times n}$ for $1 < i < n$,
- $A_{d,j} \in \mathbb{C}^{n\times 1}$.

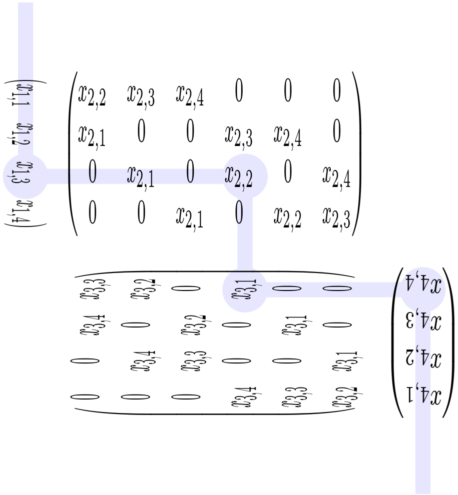
Example: $-x^3 + 3x^2y + 2xy^2 = \begin{pmatrix} x & x+y \end{pmatrix} \begin{pmatrix} y+x & 2x \\ -y & x \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix}$, hence $\mathrm{w}(-x^3 + 3x^2y + 2xy^2) \le 2$.

Larger example: $\mathrm{per}_4 = \sum_{\pi \in \mathfrak{S}_4} \prod_{i=1}^4 x_{i,\pi(i)} =$

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \end{pmatrix} \begin{pmatrix} x_{2,2} & x_{2,3} & x_{2,4} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & x_{2,3} & x_{2,4} & 0 \\ 0 & x_{2,1} & 0 & x_{2,2} & 0 & x_{2,4} \\ 0 & 0 & x_{2,1} & 0 & x_{2,2} & x_{2,3} \end{pmatrix} \begin{pmatrix} 0 & 0 & x_{3,4} & x_{3,3} \\ 0 & x_{3,4} & 0 & x_{3,2} \\ 0 & x_{3,3} & x_{3,2} & 0 \\ x_{3,4} & 0 & 0 & x_{3,1} \\ x_{3,3} & 0 & x_{3,1} & 0 \\ x_{3,2} & x_{3,1} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{4,1} \\ x_{4,2} \\ x_{4,3} \\ x_{4,4} \end{pmatrix}$$

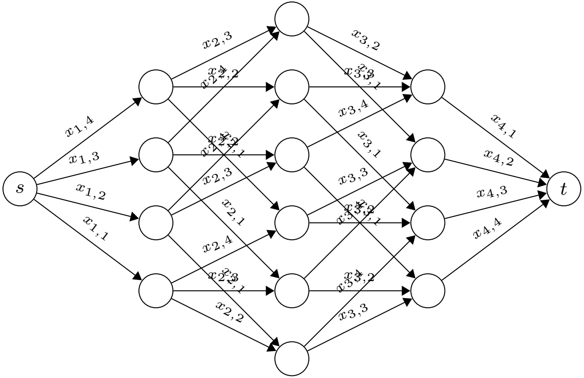
$$= \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{2,2} & x_{2,3} & x_{2,4} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & x_{2,3} & x_{2,4} & 0 \\ 0 & x_{2,1} & 0 & x_{2,2} & 0 & x_{2,4} \\ 0 & 0 & x_{2,1} & 0 & x_{2,2} & x_{2,3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & x_{3,4} & x_{3,3} & 0 & 0 \\ 0 & x_{3,4} & 0 & x_{3,2} & 0 & 0 \\ 0 & x_{3,3} & x_{3,2} & 0 & 0 & 0 \\ x_{3,4} & 0 & 0 & x_{3,1} & 0 & 0 \\ x_{3,3} & 0 & x_{3,1} & 0 & 0 & 0 \\ x_{3,2} & x_{3,1} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{4,1} \\ x_{4,2} \\ x_{4,3} \\ x_{4,4} \\ 0 \\ 0 \end{pmatrix}$$

Iterated matrix multiplication



This path corresponds to the product $x_{1,3}x_{2,2}x_{3,1}x_{4,4}$.

Iterated matrix multiplication



Algebraic branching program (ABP)

$$\sum_{s\text{-}t\text{-path }p} \prod_{e \in p} \mathrm{label}(e)$$

From this description, it is evident: $\mathrm{w}(f) \leq \mathrm{WR}(f)$.
$\mathrm{WR}(x_1^d + \cdots + x_k^d) = k$, but $\mathrm{w}(x_1^d + \cdots + x_k^d) = \lceil \frac{k}{2} \rceil$ [Kumar 2019].

Valiant's extended hypothesis

The permanent polynomial:

$$\mathrm{per}_m = \sum_{\pi \in \mathfrak{S}_m} \prod_{i=1}^{m} x_{i,\pi(i)} \quad \in \mathrm{Sym}^m \mathbb{C}^{m^2}$$

Theorem [Grenet 2011]: $\mathrm{w}(\mathrm{per}_m) \leq \binom{m}{\lceil m/2 \rceil}$

A sequence c_m is called **quasipolynomially bounded** if there exists a univariate polynomial q with $c_m \leq 2^{q(\log_2(m))}$.
(Using tilde-Landau notation: $m^{O(1)}$ is poly, and $m^{\widetilde{O}(1)}$ is quasipoly.)

Conjecture (Valiant, von zur Gathen, Toda, Malod, Portier)

$m \mapsto \mathrm{w}(\mathrm{per}_m)$ is not quasipolynomially bounded.

Polynomial, quasipolynomial, "efficient"

[Edmonds 1965] and [Cobham 1965] give arguments for "efficient = polytime":

1. All problems that can be solved efficiently seem to have a polytime algorithm
2. All computational models simulate each other with polytime overhead
3. Polytime is closed under composition

> Lots of growth behaviours are closed under composition,
> and point 1 and 2 are not true in the algebraic world!

- [Valiant 1979] has formulas, determinantal complexity, and circuits: quasipolynomial simulation overhead
- Homogenization of formulas: quasipolynomial overhead
- Homogenization of the continuant ([Dutta-Gesmundo-Ikenmeyer-Jindal-Lysikov 2023]): quasipolynomial
- Projecting metapolynomials to isotypic components [van den Berg-Dutta-Gesmundo-I-Lysikov 2024]: quasipolynomial
- Group isomorphism [Felsch-Neubüser 1970], [Rosenbaum 2012], Graph isomorphism [Babai 2015]

- [von zur Gathen 1985]:
  Some results take on a nicer form if we are somewhat more generous and allow "quasi-polynomial time".
  The notions with "qp" have nicer stability properties than "p".
- [Wigderson 2018] presentation: "$\det_n$ *is* **VP***-complete*" (under quasipoly projections).

I had a discussion with Rahul Santhanam:
A function $c : \mathbb{N} \to \mathbb{N}$ is called **fractional-exponentially bounded** if $\exists e \in \mathbb{N} : m \mapsto \underbrace{c(c(\cdots(c}_{\text{e times}}(m))\cdots))$ is exponentially bd.

- Studied in [Miltersen-Vinodchandran-Watanabe 1999].
- Seems to be a clean and more robust separating line between "efficient" and "non-efficient".

Field (in-)dependence

Sometimes techniques from $\mathbb{C}$ can be used to prove the result over arbitrary fields:

- [Forbes 2024] for [Limaye-Srinivasan-Tavenas 2021] lower bound: Evaluation of separating metapolynomial is 1.
- [McDowell-Wildon 2021] for Hermite reciprocity (metapolynomials on binary forms)

Sometimes we do not know a field-independent way:

- $\mathrm{WR}(x^p + y^p) =$
  - 2 if $\mathrm{char}(\mathbb{F}) \neq p$,
  - 1 if $\mathrm{char}(\mathbb{F}) = p$, because $x^p + y^p = (x+y)^p$.
- The matrix multiplication exponent ω can also depend on $\mathrm{char}(\mathbb{F})$.
- Modular representation theory is far from understood.

Sometimes results require a nontrivial fix. Examples for metapolynomials:

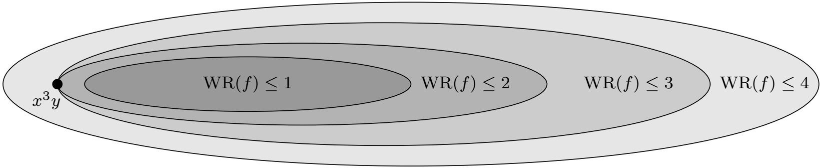
- [Kouwenhoven 1990] $\mathrm{Sym}^3\mathrm{Sym}^4\mathbb{F}^2 \not\simeq \mathrm{Sym}^4\mathrm{Sym}^3\mathbb{F}^2$ in characteristic 3 (isom. in char $= 0$: Hermite reciprocity)
- But $\mathrm{Sym}^a\mathrm{Sym}_b\mathbb{F}^2 \simeq \mathrm{Sym}_b\mathrm{Sym}^a\mathbb{F}^2$ ([Aprodu-Farkas-Papadima-Raicu-Weyman 2018], [McDowell-Wildon 2021], [Raicu-Sam 2021])

Over $\mathrm{char}(\mathbb{F}) = 2$ we have $\mathrm{per}_m = \det_m$, and there are several replacements for per_m.

Agenda

Border complexity and algebraic varieties

$12x^3y = (x+y)^4 + i^3(x+iy)^4 + i^2(x+i^2y)^4 + i(x+i^3y)^4$. Indeed, $\mathrm{WR}(x^3y) = 4$.



$\frac{3}{\varepsilon}\left((x+\varepsilon y)^4 - x^4\right) = 12x^3y + 4\varepsilon(6x^2y^2 + 4\varepsilon xy^3 + \varepsilon^2 y^4) \xrightarrow{\varepsilon \to 0} 12x^3y$

The **border Waring rank** $\underline{\mathrm{WR}}(f)$ is defined as the smallest n such that f can be approximated arbitrarily closely by polynomials of Waring rank $\leq n$. For example, $\underline{\mathrm{WR}}(x^3y) \leq 2$.

$$\{f \mid \underline{\mathrm{WR}}(f) \leq n\} = \overline{\{f \mid \mathrm{WR}(f) \leq n\}}^{\mathbb{C}} = \overline{\{f \mid \mathrm{WR}(f) \leq n\}}^{\mathrm{Zar}}$$

- Define border complexity measures for every complexity measure, for example $\underline{\mathrm{w}}$.

$$\{f \mid \underline{\mathrm{w}}(f) \leq n\} = \overline{\{f \mid \mathrm{w}(f) \leq n\}}^{\mathbb{C}} = \overline{\{f \mid \mathrm{w}(f) \leq n\}}^{\mathrm{Zar}}$$

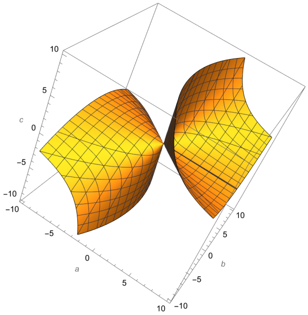
This is a general result about images of polynomial maps.

Extended Mulmuley-Sohoni Conjecture

$m \mapsto \underline{\mathrm{w}}(\mathrm{per}_m)$ is not quasipolynomially bounded.

The Zariski closure, metapolynomials

$f = ax^2 + bxy + cy^2 \in \mathrm{Sym}^2\mathbb{C}^2$ has $\mathrm{WR}(f) = 1$ if and only if $b^2 - 4ac = 0$.



$b^2 - 4ac \in \mathrm{Sym}^2(\mathrm{Sym}^2\mathbb{C}^2)$ is called a **metapolynomial**.

$\{f \mid \underline{\mathrm{WR}}(f) \le n\} = \overline{\{f \mid \mathrm{WR}(f) \le n\}}^{\mathrm{Zar}}$ simultaneous vanishing set of metapolynomials.

If $\underline{\mathrm{WR}}(h) > n$, then there exists a metapolynomial Δ that vanishes on all f with $\mathrm{WR}(f) \le n$, but $\Delta(h) \ne 0$.

$\{f \mid \underline{\mathrm{w}}(f) \le n\} = \overline{\{f \mid \mathrm{w}(f) \le n\}}^{\mathrm{Zar}}$ simultaneous vanishing set of metapolynomials.

If $\underline{\mathrm{w}}(h) > n$, then there exists a metapolynomial Δ that vanishes on all f with $\mathrm{w}(f) \le n$, but $\Delta(h) \ne 0$.

De-bordering: Do approximations significantly change the power of the computational model?

Theorem (follows from [Bini 1980]): $\lim \log_m(\mathrm{WR}(M_m)) = \omega \overset{!}{=} \lim \log_m(\underline{\mathrm{WR}}(M_m))$

[Bringmann-I-Zuiddam 2017] based on [Allender-Wang 2011]:
Theorem: width 2 affine algebraic branching programs can compute every polynomial, but only if we allow approximations

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De-bordering: [Saxena 2008], [Forbes at WACT 2016], [Bläser-Dörfler-I 2020]

$$\mathrm{w}(f) \leq \underline{\mathrm{WR}}(f)$$

More de-bordering in Pranjal Dutta's talk.
Open problems:

- Is there a sequence of degree m homogeneous polynomials f_m with $\underline{\mathrm{w}}(f_m)$ efficient, but $\mathrm{w}(f_m)$ not efficient?
- Of particular interest is $\underline{\mathrm{w}}(\mathrm{per}_m)$ vs $\mathrm{w}(\mathrm{per}_m)$: Relating Valiant's and Mulmuley-Sohoni's conjectures

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[Christandl-Hoeberechts-Nieuwboer-Vrana-Zuiddam 2024]: "Asymptotic tensor rank is characterized by polynomials"

Imprecise question: Is there some kind of "asymptotic analog" of $\underline{\mathrm{w}}$ that can be used instead of $\underline{\mathrm{w}}$ for the extended Mulmuley-Sohoni conjecture?

Non-asymptotic questions

For any **fixed degree** d, **fixed number of variables** k, **fixed complexity bound** n, consider the variety

$$X_{d,k,n} \;:=\; \{f \in \mathrm{Sym}^d\mathbb{C}^k \mid \underline{\mathrm{WR}}(f) \leq n\} \;\subseteq\; \mathrm{Sym}^d\mathbb{C}^k.$$

Theorem [Alexander-Hirschowitz 1995]

The codimension of $X_{d,k,n}$ in $\mathrm{Sym}^d\mathbb{C}^k$ is given by the naive formula $\max\{\binom{k+d-1}{d} - kn, 0\}$ that comes from counting the degrees of freedom, except when:

- $d = 2$ and $2 \leq n < k$
- $d = 3,\ k = 5,\ n = 7$
- $d = 4,\ k = 3, 4, 5,\ n = \binom{(k-1)(k+2)}{2}$

This gives rise to the notion of **generic rank** of $\mathrm{Sym}^d\mathbb{C}^k$ as the smallest n such that $X_{d,k,n} = \mathrm{Sym}^d\mathbb{C}^k$.
[Blekherman-Teitler 2014]: The maximal rank in $\mathrm{Sym}^d\mathbb{C}^k$ is at most twice the generic rank.

Open Question

Let

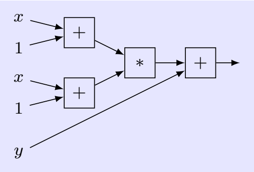
$$W_{d,k,n} \;:=\; \{f \in \mathrm{Sym}^d\mathbb{C}^k \mid \underline{\mathrm{w}}(f) \leq n\} \;\subseteq\; \mathrm{Sym}^d\mathbb{C}^k.$$

What is the codimension of $W_{d,k,n}$ in $\mathrm{Sym}^d\mathbb{C}^k$? Easier: Which $W_{d,k,n}$ are equal to $\mathrm{Sym}^d\mathbb{C}^k$?

This would give us the generic width. Blekherman-Teitler also works for $\underline{\mathrm{w}}$ (for any sub-additive measure).

To motivate the study of this question, we would like to know that $\underline{\mathrm{w}}$ is the "correct" complexity measure!

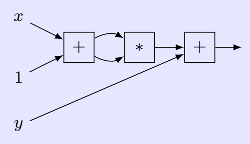
Several algebraic models of computation of quasipolynomially equivalent power



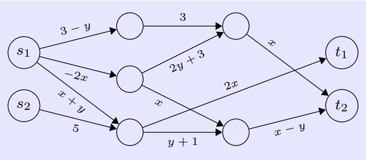
formula

$$\det\begin{pmatrix} x+1 & y \\ -1 & x+1 \end{pmatrix}$$

affine-linear determinantal presentation



circuit



algebraic branching program (ABP) = iterated matrix multiplication

- Homogeneous circuit/formula: All gates compute homogeneous polynomials.
- Input-homogeneous-linear (IHL): All inputs are homogeneous linear.

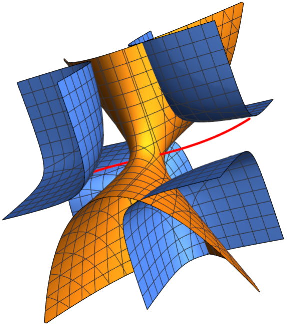
- ABP computes $\sum_i \sum_{s_i\text{-}t_i\text{-path } p} \prod_{e \in p} \text{label}(e)$
- ABP are usually **layered**.

Three homogeneous layered ABP variants:

- single-source/sink ABP: $(A_{1,1}x_1 + \ldots + A_{1,k}x_k)\cdots(A_{d,1}x_1 + \ldots + A_{d,k}x_k)$ Notation: ABP
- multi-source/sink ABP: $\mathrm{tr}((A_{1,1}x_1 + \ldots + A_{1,k}x_k)\cdots(A_{d,1}x_1 + \ldots + A_{d,k}x_k))$ Notation: tr-ABP
- $\mathrm{tr}((A_{1,1}x_1 + \ldots + A_{1,k}x_k)^d)$ Notation: trp-ABP

Irreducible varieties

An algebraic variety is called **irreducible** if it is not the finite union of proper algebraic subvarieties.



$$X_{d,k,n} = \{f \in \mathrm{Sym}^d \mathbb{C}^k \mid \underline{\mathrm{c}}(f) \leq n\}$$

😄 Irreducible:

- determinantal complexity
- size/depth of universal circuit
- size/depth of universal formula
- (tr-)ABP width
- trp-ABP width

😕 Reducible:

- circuit/formula size/depth
- (tr-)ABP number of edges/vertices
- trp-ABP number of edges/vertices

Homogeneous linear projections and orbit closures

Goal: Clean computational model in order to avoid unnatural-looking representation theory as in [Kadish-Landsberg 2013], [I-Panova 2015], [Bürgisser-I-Panova 2016].

Universal polynomials $u_{d,n}$, homogeneous of degree d, such that for all degree d homogeneous f we have

$$\mathrm{c}(f) \le n \text{ if and only if } \exists \text{ linear map } A \text{ such that } u_{d,n} \circ A = f.$$

In other words: $X_{d,k,n} = \{f \in \mathrm{Sym}^d\mathbb{C}^k \mid \underline{\mathrm{c}}(f) \le n\} = \overline{\mathsf{GL} \cdot u_{d,n}} \cap \mathrm{Sym}^d\mathbb{C}^k$

- This is stronger than just irreducible.
- Unlike Valiant's projections, we only allow linear maps, not affine-linear maps!

ABPs and tr-ABPs and and trp-ABPs give universal polynomials:

- $\mathrm{IMM}_{n,d} = \left(x_{i,j}^{(1)}\right)_{1\times n} \cdot \left(x_{i,j}^{(2)}\right)_{n\times n} \cdots \left(x_{i,j}^{(d-1)}\right)_{n\times n} \cdot \left(x_{i,j}^{(d)}\right)_{n\times 1} \in \mathrm{Sym}^d\mathbb{C}^{n^2(d-2)+2n}$
- $\text{tr-IMM}_{n,d} = \mathrm{tr}\left(\left(x_{i,j}^{(1)}\right)_{n\times n} \cdots \left(x_{i,j}^{(d)}\right)_{n\times n}\right) \in \mathrm{Sym}^d\mathbb{C}^{n^2 d}$
- $\mathrm{trp}_{n,d} = \mathrm{tr}\left(\left(x_{i,j}\right)_{n\times n}^{d}\right) \in \mathrm{Sym}^d\mathbb{C}^{n^2}$

- Using the determinant like Valiant is not possible, because it has only one parameter.
- The original [Mulmuley-Sohoni] contain universal circuits, but these constructions do not work.
  Also, it is unclear how to model concepts such as "constant depth circuits".

Newton's identities, determinant, characteristic polynomial, trp

- Define $\mathrm{charpoly}_{n,d}$ via $\sum_{d=0}^{n} \mathrm{charpoly}_{n,n-d} \cdot t^d = \det(X + t \cdot I)$.
- $\mathrm{charpoly}_{n,n} = \det_n$

Robert Andrews: $\mathrm{charpoly}_{n,d}$ gives a universal polynomial, same complexity as $\mathrm{IMM}_{n,d}$, $\text{tr-IMM}_{n,d}$, $\mathrm{trp}_{n,d}$

Newton's identities:

$$j \cdot e_j = \sum_{i=1}^{j} (-1)^{i-1} \cdot e_{j-i} \cdot p_i$$
$$j \cdot \mathrm{charpoly}_{n,j} = \sum_{i=1}^{j} (-1)^{i-1} \cdot \mathrm{charpoly}_{n,j-i} \cdot \mathrm{trp}_{n,i}$$

☹ $\mathrm{trp}_{n,d}$ and $\mathrm{charpoly}_{n,d}$ are not characterized by their stabilizers.

[Dutta-Gesmundo-I-Jindal-Lysikov 2023]:
This variant of the continuant polynomial is also universal and of the same complexity:

$$C_{n,d} = \sum_{\substack{1 \leq i_1 < \cdots < i_d \leq n \\ i_j \equiv j \pmod 2}} x_{i_1} x_{i_2} \cdots x_{i_d} \qquad \in \mathrm{Sym}^d \mathbb{C}^n$$

☹ $C_{n,d}$ is not characterized by its stabilizer.

Open problem: Can we get an easier polynomial that is still characterized by its stabilizer?
Open problem: Is the elementary symmetric polynomial universal for efficient computation? (see [Shpilka 2002])

Only two candidates left: IMM width vs tr-IMM width

Both are characterized by their stabilizer, which is reductive.
Both tangent spaces can be described in terms of flow vector space.
Both are polystable in their respective spaces.
Both noncommutative variants are structure tensors:

- IMM for iterated product of matrices with a vector
- $\mathrm{tr\text{-}IMM}$ for iterated product of matrices

Both noncommutative variants are closely related to tensor networks:

- IMM for the path graph
- $\mathrm{tr\text{-}IMM}$ for the circle graph

$\mathrm{tr\text{-}IMM}$ has a cyclic group in its stabilizer, which IMM does not have.

Advantages of IMM over $\mathrm{tr\text{-}IMM}$:

- For IMM, for the noncomm. variant the set $X_{d,k,n} = \{t \in \bigotimes^d \mathbb{C}^k \mid \text{nc-w}(t) \leq k\}$ is Zariski-closed! [Nisan 1990]
  For $\mathrm{tr\text{-}IMM}$ it is **not** Zariski-closed! [Landsberg-Qi-Ye 2011]
- Grenet's algorithm for the noncommutative permanent is optimal for noncommutative IMM.
- The codimensions are as expected for noncommutative IMM [Haegeman-Mariën-Osborne-Verstraete 2014], but we do not know this for noncommutative $\mathrm{tr\text{-}IMM}$.
- [Bürgisser-Cucker-Lairez 2023] use IMM to “introduce a universal model of random polynomial systems with prescribed evaluation complexity L.”

All evidence towards: $\mathrm{IMM}_{n,d}$ (i.e., homogeneous ABP width) is the cleanest computational model. 😄

Open problem: Can we find something easier with the same properties?

Agenda

Search space restriction to isotypic metapolynomials

- Elements of $\mathrm{Sym}^\delta \mathrm{Sym}^d \mathbb{C}^k$ are called **metapolynomials**.
- $X_{d,k,n} := \{f \in \mathrm{Sym}^d \mathbb{C}^k \mid \underline{\mathrm{c}}(f) \le n\}$.

$X_{d,k,n}$ is closed under the action of GL_k, for example, $\underline{\mathrm{w}}(f) = k$ iff $\underline{\mathrm{w}}(gf) = k$ for $g \in \mathsf{GL}_k$.
The action is "pulled back" to a linear action on metapolynomials: $(g\Delta)(f) := \Delta(g^t f)$.

Restriction to isotypic Δ

Every GL_k-representation decomposes uniquely into a direct sum of isotypic components, for example:

$$\mathrm{Sym}^\delta \mathrm{Sym}^d \mathbb{C}^k = \bigoplus_{\lambda \vdash k} (\mathrm{Sym}^\delta \mathrm{Sym}^d \mathbb{C}^k)^\lambda.$$

Proposition: If Δ vanishes on $X_{d,k,n}$ and not vanish on h, then one of its isotypic components also does.

Theorem [Bläser-Christandl-Zuiddam 2018] based on [Hauenstein-I-Landsberg 2013]

The border support rank of the 2×2 matrix multiplication tensor is 7.

Proof idea (overly simplified):

- Take a degree $\delta = 20$ metapolynomial Δ that vanishes on all border rank 6 tensors
- Show that it does not vanish on ANY tensor whose supoprt is 2×2 matrix multiplication

- dimension of $\mathbb{C}[x_1, \ldots, x_{64}]_{20} = 817980867927266 4720$.
- dimension of $(5,5,5,5)^3$-isotypic polynomials in that space $= 4$.

Complexity of isotypic metapolynomials

$$\begin{aligned} c_{2,0,0}c_{0,2,0}c_{0,0,2} &= \tfrac{1}{30}(c_{1,0,1}^2 c_{0,2,0} + 2c_{1,1,0}c_{1,0,1}c_{0,1,1} + c_{2,0,0}c_{0,1,1}^2 + c_{1,1,0}^2 c_{0,0,2} + 2c_{2,0,0}c_{0,2,0}c_{0,0,2}) \\ &+ \tfrac{1}{20}(c_{0,2,0}c_{1,0,1}^2 - 3c_{0,1,1}c_{1,0,1}c_{1,1,0} + c_{0,0,2}c_{1,1,0}^2 + c_{0,1,1}^2 c_{2,0,0} + 12c_{0,0,2}c_{0,2,0}c_{2,0,0}) \\ &+ \tfrac{1}{12}(-c_{1,0,1}^2 c_{0,2,0} + c_{1,1,0}c_{1,0,1}c_{0,1,1} - c_{2,0,0}c_{0,1,1}^2 - c_{1,1,0}^2 c_{0,0,2} + 4c_{2,0,0}c_{0,2,0}c_{0,0,2}) \end{aligned}$$

Can the isotypic components Δ^λ of Δ have much higher circuit complexity than Δ?

[van den Berg-Dutta-Gesmundo-I-Lysikov 2024]:

- $\mathrm{w}(\Delta^\lambda)$ is quasipolynomial in $\mathrm{w}(\Delta)$
- Lower bounds can be proved w.l.o.g. via isotypic metapolynomials.
- Algebraic natural proofs can be assumed to be isotypic.
- With a bit more work this resolves Open Question 2 in [Grochow-Kumar-Saks-Saraf 2017].

Multiplicities

- $X_{d,k,n} := \{f \in \mathrm{Sym}^d\mathbb{C}^k \mid \underline{\mathrm{c}}(f) \le n\}$.

Since $X_{d,k,n}$ is closed under the action of GL_k:

- The vanishing ideal $I(X_{d,k,n})$ in each degree δ is a GL_k-representation.
- The coordinate ring $\mathbb{C}[X_{d,k,n}]$ in each degree δ is a GL_k-representation: $\mathbb{C}[X_{d,k,n}]_\delta = \mathrm{Sym}^\delta\mathrm{Sym}^d\mathbb{C}^k/I(X_{d,k,n})_\delta$.

$$\underline{\mathrm{w}}(f) \le n \iff f \in X_{d,k,n} \iff \overline{\mathsf{GL}_k f} \subseteq X_{d,k,n} \implies \underbrace{\mathbb{C}[X_{d,k,n}]_\delta}_{\bigoplus_\lambda (S_\lambda(\mathbb{C}^k))^{\oplus z_\lambda}} \twoheadrightarrow \underbrace{\mathbb{C}[\overline{\mathsf{GL}_k f}]_\delta}_{\bigoplus_\lambda (S_\lambda(\mathbb{C}^k))^{\oplus y_\lambda}} \implies \forall\lambda : z_\lambda \ge y_\lambda$$

If $\exists\lambda :\ z_\lambda < y_\lambda$, then $\underline{\mathrm{w}}(f) > n$.

[Mulmuley-Sonohi 2001, 2008] had hopes for **occurrence obstructions**: $z_\lambda = 0 < y_\lambda$, but the situation is unclear: [I-Panova 2015], [Bürgisser-I-Panova 2016], [Dörfler-I-Panova 2019]

Hope for multiplicities

Definition (characterized by stabilizer)

A point $f \in V$ is called **characterized by its stabilizer** H, if $\dim(V^H) = 1$.

Many points are characterized by their stabilizer: $\mathrm{IMM}_{n,d}$, $\text{tr-IMM}_{n,d}$, $\det_n$, per_m.

Proposition

If f is characterized by its stabilizer, each of the following pieces of information is sufficient to determine the others:
(1) The orbit Gf (2) The orbit closure $\overline{Gf}$ (3) The stabilizer of f up to conjugation

Polystable: $\mathrm{IMM}_{n,d}$, $\text{tr-IMM}_{n,d}$, $\det_n$, per_m.

Proposition

For a polystable f, under technical assumptions (G must be be a compact Lie group and the stabilizer H must be connected and $\mathbb{C}^m$ must be an irred. H-representation) we can enlarge this list [Yu 2016], based on [Larsen-Pink 1990]:

(4) The multiplicities in the coordinate ring $\mathbb{C}[Gf]$

(5) The multiplicities in the coordinate ring $\mathbb{C}[\overline{Gf}]$

Theorem [I-Kandasamy 2020]: For $\lambda = (4k, 2k, 2k, \ldots, 2k)$ we have
$\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_k(x_1^k + \cdots + x_k^k)}]) > \mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathsf{GL}_k(x_1 \cdots x_k)}])$ and these obstructions are obtained **from the stabilizers**.

Open problem: If two polystable points f and h are characterized by their stabilizer, and $\overline{Gh} \not\subseteq \overline{Gf}$, does there always exist λ with $\mathrm{mult}_\lambda(\mathbb{C}[\overline{Gh}]) > \mathrm{mult}_\lambda(\mathbb{C}[\overline{Gf}])$? If not in general, then under which conditions?

Plethysm and Kronecker coefficients

$$\mathrm{Sym}^\delta \mathrm{Sym}^d \mathbb{C}^k \simeq \bigoplus_{\lambda \vdash_k \delta d} (S_\lambda \mathbb{C}^k)^{\oplus a_\lambda(\delta[d])}$$

- The multiplicities $a_\lambda(\delta[d])$ are called **plethysm coefficients**.
- $(\lambda, \delta, d) \mapsto a_\lambda(\delta[d])$ is in **GapP**. Open problem: Is it in **#P**? [Stanley 2000 Problem 9]
- Deciding positivity is **#P**-hard [Fischer-I 2020].

$$\mathrm{Sym}^\delta(\mathbb{C}^k \otimes \mathbb{C}^k \otimes \mathbb{C}^k) \simeq \bigoplus_{\lambda,\mu,\nu} (S_\lambda \mathbb{C}^k \otimes S_\mu \mathbb{C}^k \otimes S_\nu \mathbb{C}^k)^{\oplus k(\lambda,\mu,\nu)}$$

- The **Kronecker coefficient** $k(\lambda, \mu, \nu)$.
- $(\lambda, \mu, \nu) \mapsto k(\lambda, \mu, \nu)$ is in **GapP**. Open problem: Is it in **#P**? [Stanley 2000 Problem 10]
- Deciding positivity is **#P**-hard [I-Mulmuley-Walter 2015].

Proved **via GCT** [I-Panova 2015]: For every partition λ with $d \geq \sum_{i\geq 2} \lambda_i$, $\delta \geq 2$ we have $k(\lambda, d^\delta, d^\delta) \geq a_\lambda(\delta[d])$.

[I-Panova 2024]: W.l.o.g. we can assume all Kronecker coefficient parameters λ, μ, ν to have long first rows.

[Krillov 2004] conjectured that such Kronecker coefficients are in **#P**. We prove equivalence to [Stanley 2000, problem 10].

Open problem: Does a similar result hold for the plethysm coefficients, i.e., can we assume that λ has a long first row?

What is in #P and what is not? Are our multiplicities in #P?

☹:

- [I-Pak 2022]: Techniques to prove non-containment in **#P**, based on [Hertrampf-Vollmer-Wagner 1995], and developed further for finite automata in [Dörfler-I 2024].
  Open problem: What is the analogue of [Dörfler-I 2024] for pushdown automata?
- Representation theoretic quantities are sometimes not in **#P**:
  [I-Pak-Panova 2022] prove that if the square of the character of the symmetric group is in **#P**, then $\mathbf{PH} = \Sigma_2^{\mathbf{p}}$.
- More combinatorics results in this direction in [Chan-Pak 2024].

😄:

- [I-Subramanian 2023] (based on [Bravyi-Chowdhury-Gosset-Havlíček-Zhu 2023]) Kronecker and plethysm coefficients are in **#BQP**. Using known techniques to prove them outside **#P** would imply new results on quantum nondeterminism.
- Quantum group approach: [Blasiak-Mulmuley-Sohoni 2007], [Mulmuley 2007], [Adsul-Sohoni-Subrahmanyam 2010], get Kashiwara crystals for several $k(\lambda, \mu, \nu)$

Summary

- Homogeneous single source/sink ABP width seems to be the cleanest computational model.
- Metapolynomials are used for proving border complexity lower bounds. W.l.o.g. they are isotypic.
- Multiplicities can sometimes be obtained from the stabilizers without constructing metapolynomials explicitly.
- Can we get enough information about our multiplicities? Are they in $\#\mathbf{P}$?

Thank you for your attention!