Generic Border Subrank

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Notations

- K: algebraic closed field
- $V := (K^n)^{\otimes d}$

For $r \in \mathbb{Z}_{>0}$, define the **unit (diagonal) tensor** of size r

$$I_r := \sum_{i=1}^{r} (e_i)^{\otimes d} \in (K^r)^{\otimes d}$$

where $\{e_i\}_{i=1}^r$ is the standard basis of K^r.

Subrank is a measure of how much a tensor can be diagonalized.

Ranks of tensors

For $T \in (K^n)^{\otimes d}$ and $S \in (K^m)^{\otimes d}$, we say $S \leq T$ (S is a restriction of T) if

$$S = (\varphi_1, \ldots, \varphi_d) \cdot T \text{ for some linear maps } \varphi_i : K^n \to K^m.$$

Definition (Rank, Border Rank, Subrank, Border Subrank)

Given $T \in K^n \otimes \cdots \otimes K^n$,

$$\mathbf{R}(T) := \min\{r : T \leq I_r\},\ \underline{\mathbf{R}}(T) := \min\{r : T \in \overline{\{S : S \leq I_r\}}\}$$
$$\mathbf{Q}(T) := \max\{r : I_r \leq T\},\ \underline{\mathbf{Q}}(T) := \max\{r : I_r \in \overline{\{S : S \leq T\}}\}.$$

- For $d = 2$, all ranks = matrix rank.
- $\mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T) \leq \underline{\mathbf{R}}(T) \leq \mathbf{R}(T)$ and $0 \leq \mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T) \leq n$.
- If $S \leq T$, then $\mathbf{Q}(S) \leq \mathbf{Q}(T)$. Same for $\mathbf{R}, \underline{\mathbf{R}}, \underline{\mathbf{Q}}$.
- $\underline{\mathbf{Q}}(T) = \max\{r \leq n : I_r \in \overline{G \cdot T}\}$, where $G = \mathrm{GL}_n(K) \times \cdots \times \mathrm{GL}_n(K)$

Motivation from Complexity Theory

The *exponent of matrix multiplication* is defined as

$$\omega := \inf\{h \in \mathbb{R} : \mathbf{R}(M_{\langle m,m,m\rangle}) = \mathcal{O}(m^h)\},$$

where $M_{\langle m,m,m\rangle}$ is the $m \times m \times m$ matrix multiplication tensor.

- $2 \leq \omega \leq \log_2 7 < 2.81 < 3$ [Strassen]
- A well-known method to find upper bounds on ω is the *laser method*, which utilizes an intermediate tensor T.
- The *laser method* made steady progress until 1990 where the estimate went down to around 2.376.
- Best bound: $\omega < 2.371339$ [Alman, Duan, Williams, Xu, Xu, Zhou, 2025]
- The intermediate tensor used faces a barrier which is related to **subrank**!

Why **subrank**?

The intermediate tensor T needed in the *laser method* has this property:

$$\bigoplus_i M_{\langle m_i, m_i, m_i \rangle} \leq T^{\boxtimes N}.$$

- $\sum_i m_i^{\omega} \leq \underset{\sim}{\mathbf{R}}(\bigoplus_i M_{\langle m_i, m_i, m_i \rangle}) \leq \underset{\sim}{\mathbf{R}}(T)^N$ [Schönhage, 1982]
- $\sum_i m_i^{2} \leq \mathbf{Q}(\bigoplus_i M_{\langle m_i, m_i, m_i \rangle}) \leq \mathbf{Q}(T^{\boxtimes N})$

If the subrank of the tensor we use is small, then m_i will be small.

In that case, we cannot get good upper bounds on ω.

The intermediate tensors of large subrank are good to get bounds for ω!

Example subrank $\neq$ border subrank

Let $T = I_3 + e_1 \otimes e_2 \otimes e_3 \in K^3 \otimes K^3 \otimes K^3$.

$\mathbf{Q}(T) = 2$:

$$\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \right) \cdot T = I_2$$

Since $T \ncong I_3$, $\mathbf{Q}(T) \neq 3$.

$\underline{\mathbf{Q}}(T) = 3$:

$$\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix} \right) \cdot T$$

$$= I_3 + te_1 \otimes e_2 \otimes e_3 \to I_3 \text{ as } t \to 0$$

$$\implies I_3 \in \overline{G \cdot T} \implies \underline{\mathbf{Q}}(T) = 3$$

Generic Subrank

For a generic tensor $T \in (\mathbb{C}^n)^{\otimes 3}$, $\mathbf{R}(T) = \underline{\mathbf{R}}(T) =$ maximum border rank $\sim n^2/3$.

$\mathbf{Q}(T)$ and $\underline{\mathbf{Q}}(T)$ have different behavior.

Theorem (Derksen, Makam, Zuiddam, 2022)

1. *There is a Zariski open subset $U \subset V$ and integer r such that for all $T \in U$, $\mathbf{Q}(T) = r$. Call this r generic subrank, denoted $\mathbf{Q}(\underbrace{n, \ldots, n}_{d})$.*
2. $\mathbf{Q}(\underbrace{n, \ldots, n}_{d}) = \Theta(n^{1/(d-1)})$ *for $n \to \infty$.*
3. *For $d = 3$, $3\lfloor \sqrt{n/3 + 1/4} - 1/2 \rfloor \le \mathbf{Q}(n, n, n) \le \lfloor \sqrt{3n - 2} \rfloor$*

Proposition (Gesmundo, 2022)

$\underline{\mathbf{Q}}(n, n, n) \le n - 1$ *for $n \ge 3$.*

Generic Border Subrank

Proposition

There is a Zariski open subset $U \subset V$ and integer r such that for all $T \in U$, $\underline{\mathbf{Q}}(T) = r$. Call this r generic border subrank, denoted $\underline{\mathbf{Q}}(\underbrace{n, \ldots, n}_{d})$.

The generic border subrank has the same growth rate as generic subrank:

Main Theorem (Baiggi, C., Draisma, Rupniewski, 2024)

$\underline{\mathbf{Q}}(\underbrace{n, \ldots, n}_{d}) = \Theta(n^{1/(d-1)})$ *for* $n \to \infty$.

However, for $d = 3$ and n sufficiently large, the generic border subrank is greater than the generic subrank:

Theorem (Baiggi, C., Draisma, Rupniewski, 2024)

$\underline{\mathbf{Q}}(n, n, n) \geq \lfloor \sqrt{4n} \rfloor - 3.$

Sketch proof of the Main Theorem

To prove $\underline{\mathbf{Q}}(\underbrace{n, \ldots, n}_{d}) = \Theta(n^{1/(d-1)})$ for $n \to \infty$, we need:

Lower Bound: There is some constant C_l such that

$$\underline{\mathbf{Q}}(n, \ldots, n) \geq C_l \cdot n^{1/(d-1)} \text{ for } n \text{ large enough.}$$

This can be proved by $\mathbf{Q}(n, \ldots, n) = \Theta(n^{1/(d-1)})$ and $\mathbf{Q} \leq \underline{\mathbf{Q}}$.

Upper Bound: There is some constant C_u such that

$$\underline{\mathbf{Q}}(n, \ldots, n) \leq C_u \cdot n^{1/(d-1)} \text{ for } n \text{ large enough.}$$

To prove this, we estimate $\dim\{T \in V : \underline{\mathbf{Q}}(T) \geq r\}$.

Generalised Hilbert-Mumford Criterion

$\underline{\mathbf{Q}}(T) \geq r \iff I_r \in \overline{G \cdot T}$

Proposition (Hilbert-Mumford Criterion)

If a connected, reductive, algebraic group H acts on an affine variety Z, $p, q \in Z$ satisfy $q \in \overline{H \cdot p}$, and $H \cdot q$ is closed, then there is a one-parameter subgroup (group homomorphism) $\lambda : K^\times \to H$ such that

$$\lim_{t \to 0} \lambda(t) \cdot p \in H \cdot q.$$

But $G \cdot I_r$ is not closed.

Proposition (Generalised Hilbert-Mumford Criterion)

If a connected, reductive, algebraic group H acts on an affine variety Z and $p, q \in Z$ satisfy $q \in \overline{H \cdot p}$, then there is a point $\tilde{q} \in H \cdot q$ and a one-parameter subgroup $\lambda : K^\times \to H$ such that

$$\lim_{t \to 0} \lambda(t) \cdot p = \lim_{t \to \infty} \lambda(t) \cdot \tilde{q}, \text{ and both limits exist.}$$

Thank you!